Geometric Interpretation of Eigenvectors

Example: The eigenvalues of the matrix $A = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$.

1. Find an eigenvector $\boldsymbol{v}_1$ with corresponding eigenvalue $\lambda_1 = 1$. Sketch the vectors $\boldsymbol{v}_1$ and $A\boldsymbol{v}_1$ head-to-tail. What do you observe?

$E_{\lambda_1} = null(A - \lambda_1 I) = null\left(\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right) = null\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)$

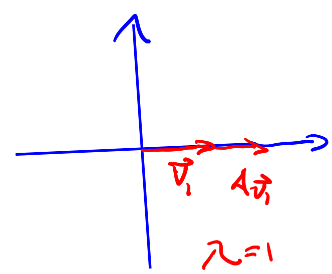
$= \left\{ \begin{bmatrix}x_1\\ x_2\end{bmatrix} : x_2 = 0 \right\} = \left\{ t\begin{bmatrix}1\\ 0\end{bmatrix} : t \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right).$

$v_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$

$A\vec{v}_1 = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$

$A\vec{x} = \lambda\vec{x}, \ \vec{x} \neq \vec{0}$

$\vec{x}, A\vec{x}$ are parallel



2. Find an eigenvector $\boldsymbol{v}_2$ with corresponding eigenvalue $\lambda_2 = 2$. Sketch the vectors $\boldsymbol{v}_2$ and $A\boldsymbol{v}_2$ head-to-tail. What do you observe?

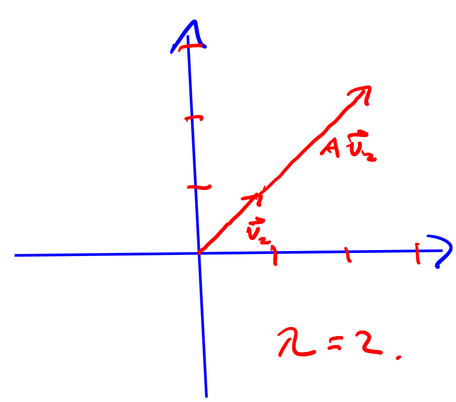
$E_{\lambda_2} = null(A - 2I) = null\left(\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}\right) = null\left(\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\right)$

$= \left\{ \begin{bmatrix}x_1\\ x_2\end{bmatrix} : x_1 - x_2 = 0 \right\} = \left\{ t\begin{bmatrix}1\\ 1\end{bmatrix} : t \text{ in } \mathbb{R} \right\} = span\left(\begin{bmatrix}1\\ 1\end{bmatrix}\right).$

$\vec{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$

$A\vec{v}_2 = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}2\\ 2\end{bmatrix}$

$= 2\begin{bmatrix}1\\ 1\end{bmatrix}.$



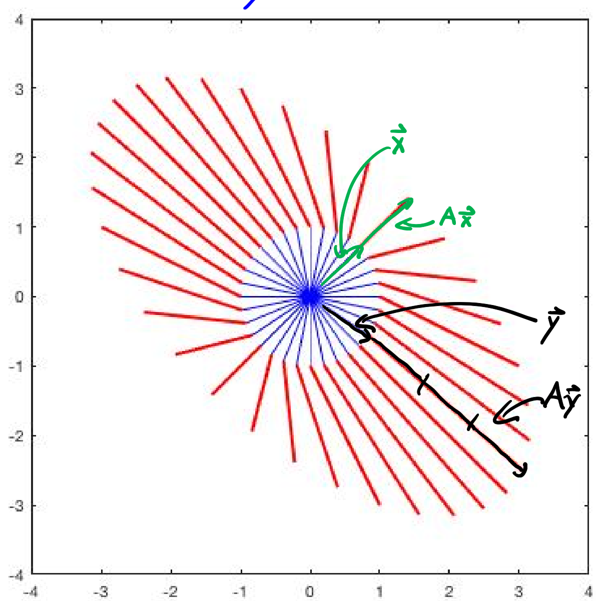
Example: The following figure shows the action of a certain 2×2 mystery matrix on a set of input vectors. The thin blue lines are input vectors $\boldsymbol{v}$ and the thicker red lines are the corresponding output vectors $A\boldsymbol{v}$ drawn head-to-tail. Use the figure to find two linearly independent eigenvectors of A as well as the corresponding eigenvalues.

$A\vec{x} = \lambda \vec{x}, \ \vec{x} \neq 0$

$A\vec{x} = \underline{1}\, \vec{x}$

$\lambda_1 = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$A\vec{y} = \underline{3}\, \vec{y}$

$\lambda_2 = 3$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Use the fact that the mystery matrix is $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ to check your answers!

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad A\vec{v}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\vec{v}_1$ ✓

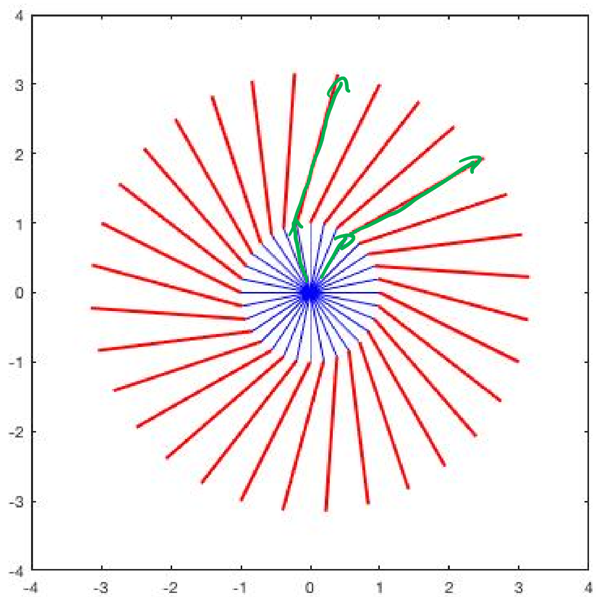
$\lambda_1 = 1$ ✓

$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad A\vec{v}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3\vec{v}_2$ ✓

$\lambda_2 = 3$ ✓

Example: The following figure shows the action of a certain 2×2 matrix on a set of input vectors. The thin blue lines are input vectors $\boldsymbol{v}$ and the thicker red lines are the corresponding output vectors $A\boldsymbol{v}$ drawn head-to-tail. After consulting the following figure, what can you say about the eigenvalues of A?

$A\vec{x} = \lambda\vec{x}, \ \vec{x} \neq \vec{0}$



No "real" eigenvalues.

The mystery matrix is $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Try and calculate the eigenvalues of A. Observations?

quadratic equ

$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 + 1 = \lambda^2 - 4\lambda + 5$

$\lambda = \dfrac{4 \pm \sqrt{16-20}}{2} = \dfrac{4 \pm \sqrt{-4}}{2}$